Technische
Universität
Braunschweig

Time-adaptive methods for the incompressible Navier-Stokes equations

Joachim Rang, Thorsten Grahs, Justin Wiegmann, 29.09.2016

Contents

Semi-Discretisation in space

- Discretise incompressible Navier–Stokes equations in space with Finite differences, volumes or elements.
- MOL-DAE:

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix} - \begin{pmatrix} A(\mathbf{u}) & B \\ B^\top & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}$$

  with M ... mass matrix (regular) and A stiffness matrix
- Assumption: $B^\top M^{-1} B$ is regular
- Then: MOL-DAE has differentiation index 2.

Initial pressure

Semi-discretised problem:

$$M\dot{\mathbf{u}} = \mathbf{f} - A(\mathbf{u})\mathbf{u} - B\mathbf{p}$$

$$B^\top \mathbf{u} = 0$$

Differentiation: $B^\top \dot{\mathbf{u}} = 0$

It follows

$$B^\top \dot{\mathbf{u}} = B^\top M^{-1}(\mathbf{f} - A(\mathbf{u})\mathbf{u} - B\mathbf{p}) = 0$$

and for $\mathbf{u} = \mathbf{u}_0$ and $\mathbf{p} = \mathbf{p}_0$

$$B^\top M^{-1}(\mathbf{f} - A(\mathbf{u}_0)\mathbf{u}_0) = B^\top M^{-1} B\mathbf{p}_0$$

Properties

- problems to be solved: stiff ODEs, PDEs, DAEs
- sufficiently high order
- no order reduction for the Prothero–Robinson example
- good stability properties (at least strongly A-stable)
- adaptive timestep control without further computations
- “low” costs for the linear algebra

Examples:

- Diagonally implicit Runge–Kutta methods (special cases: Euler, CN, FS)
- linear implicit Runge–Kutta methods (Rosenbrock–Wanner methods)
- fully implicit Runge–Kutta methods with a regular coefficient matrix (for example Radau-IIA methods)

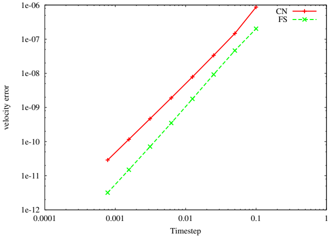
Motivating example

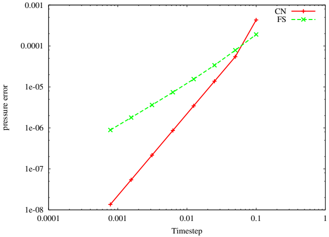
Let $T = 1$, $\Omega = (0,1)^2$ and $Re = 1$.

Exact solution of NSE:

$$u_1(t,x,y) = (y^2 + x)\sin t, \quad u_2(t,x,y) = (x^2 - y)\sin t,$$
$$p(t,x,y) = e^{-t}(x + y - 1).$$

$\tau = \frac{1}{10 \cdot 2^N}$ with $N = 0, 1, \ldots, 8$



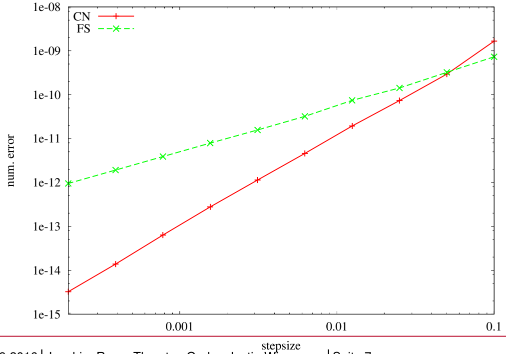


Example of Prothero and Robinson

$$\dot{u} = \lambda(u(t) - \varphi(t)) + \dot{\varphi}(t), \quad u(0) = \varphi(0), \qquad \lambda = -10^6, t \in (0, 1/10)$$

with

$$\varphi(t) = \sin\left(t + \frac{\pi}{4}\right).$$



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- **Summary and Outlook**

Runge-Kutta-methods

$$M\dot{\mathbf{u}} = \mathbf{F}(t, \mathbf{u}), \qquad \mathbf{u}(0) = \mathbf{u}_0, \tag{1}$$

M ... mass-matrix. Let $s \in \mathbb{N}$. The one-step-method

$$M\mathbf{k}_i = \mathbf{F}\left(t_m + c_i\tau, \mathbf{u}_m + \tau \sum_{j=1}^{s} a_{ij}\mathbf{k}_j\right)$$

$$\mathbf{u}_{m+1} = \mathbf{u}_m + \tau \sum_{i=1}^{s} b_i\mathbf{k}_i$$

is called s-stage Runge-Kutta-method (RK-method).

- a_{ij}, b_i and c_i ... coefficients of the method
- one nonlinear system of dimension ns

Adaptive timestep control

- 1. RK method (order p):

$$\mathbf{u}_{m+1} = \mathbf{u}_m + \tau \sum_{i=1}^{s} b_i \mathbf{k}_i$$

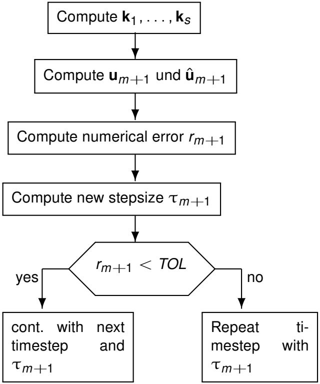
- 2. RK method (order $p-1$):

$$\hat{\mathbf{u}}_{m+1} = \mathbf{u}_m + \tau \sum_{i=1}^{s} \hat{b}_i \mathbf{k}_i$$

- TOL ... given tolerance
- PI-controller:

$$\tau_{m+1} = \rho \frac{\tau_m^2}{\tau_{m-1}} \left(\frac{TOL \cdot r_m}{r_{m+1}^2} \right)^{1/p}, \text{ with}$$

$$r_m = \|\mathbf{u}_m - \hat{\mathbf{u}}_m\|$$



Simplifying conditions

$$B(p): \quad \sum_{i=1}^{s} b_i c_i^{k-1} = 1/k \qquad k = 1, \ldots, p,$$

$$C(q): \quad \sum_{j=1}^{s} a_{ij} c_j^{k-1} = c_i^k/k \qquad i = 1, \ldots, s, k = 1, \ldots, q,$$

$$D(r): \quad \sum_{i=1}^{s} b_i c_i^{k-1} a_{ij} = b_j(1 - c_j^k)/k \qquad j = 1, \ldots, s, k = 1, \ldots, r.$$

An RK-method with s internal stages has convergence order p, if the simplifying conditions $B(p)$, $C(l)$, and $D(m)$ with

$$p \leqslant \min\{l + m + 1, 2l + 2\}$$

are satisfied.

DIRK methods

diagonally implicit Runge–Kutta (DIRK) method: $a_{ij} = 0$ for $i < j$ and $a_{ii} \neq 0$ for $i = 2, \dots, s$

Advantage: "only" s nonlinear system of dimension n instead of one nonlinear system of dimension ns

But:

- DIRK method with $a_{11} \neq 0$: maximal stage order 1
- DIRK method with $a_{11} = 0$: maximal stage order 2

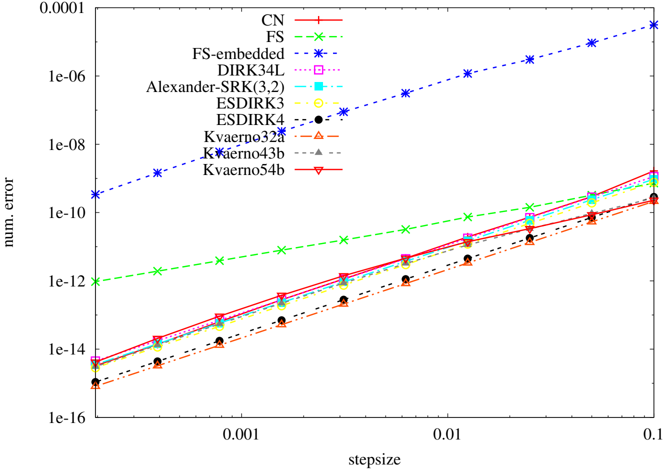
stiffly accurate: $a_{si} = b_i$, $i = 1, \dots, s$, $c_s = 1$

Consider only stiffly accurate methods with $q = 2$ and $\gamma := a_{ii}$, $i = 1, \dots, s \quad \Longrightarrow \quad$ ESDIRK methods

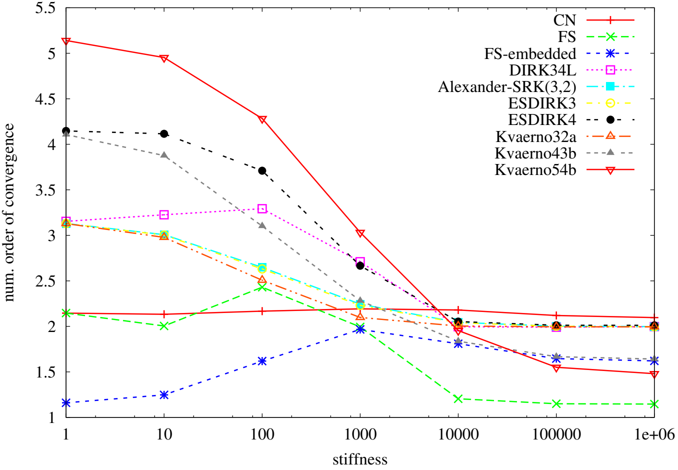
Studied methods

- CN: $s = p = q = 2$, A–stable
- FS: $s = 4$, $p = 2$, $q = 1$, strongly A–stable
- DIRK34L: $s = 4$, $p = 3$, $q = 2$, L-stable
- Alexander-SRK(3,2): $s = 4$, $p = 3$, $q = 2$, L-stable (Alexander 2003)
- ESDIRK3: $s = 4$, $p = 3$, $q = 2$, L-stable (Bijl, Carpenter, Vatsa, and Kennedy2002)
- ESDIRK4: $s = 6$, $p = 4$, $q = 2$, L-stable (Bijl, Carpenter, Vatsa, and Kennedy 2002)
- Kvaerno32a: $s = 4$, $p = 3$, $q = 2$, L-stable (Kværnø 2004)
- Kvaerno43b: $s = 5$, $p = 4$, $q = 2$, L-stable (Kværnø 2004)
- Kvaerno54b: $s = 7$, $p = 5$, $q = 2$, L-stable (Kværnø 2004)

Example of Prothero and Robinson



Numerical order of convergence



Local error

- Apply an ESDIRK method to the ODE of Prothero–Robinson
- Local error:

$$
\begin{aligned}
&\delta_\tau(t_{m+1}) \\
&= \tau[b_1 + \tilde{\mathbf{b}}^\top \tilde{A}^{-1}(\tilde{\mathbf{c}} - \mathbf{a}_1) - 1]\dot{\varphi}_m \\
&+ \sum_{k=2}^{p} \left[\tilde{\mathbf{b}}^\top \tilde{A}^{-1} \tilde{\mathbf{c}}^k - 1\right] \varphi_m^{(k)}] \frac{\tau^k}{k!} + \mathcal{O}(\tau^{p+1}) \\
&+ \sum_{k=2}^{\infty} \tilde{\mathbf{b}}^\top \left[\tilde{A}^{-k}(\tilde{\mathbf{c}} - \mathbf{a}_1) - \tilde{A}^{-k+1}\tilde{\mathbf{e}}\right] \dot{\varphi}_m \frac{\tau}{z^{k-1}} \\
&- \sum_{k=2}^{\infty} \tilde{\mathbf{b}}^\top \sum_{l=1}^{k-2} \tilde{A}^{-l} \left[\tilde{A}^{-1}\tilde{\mathbf{c}}^{k-l} - (k-l)\tilde{\mathbf{c}}^{k-l-1}\right] \varphi_m^{(k-l)} \frac{\tau^{k-l}}{(k-l)!\,z^l}.
\end{aligned}
$$

Order conditions

Finally we get the new order conditions

$$
\begin{aligned}
b_1 + \tilde{\mathbf{b}}^\top \tilde{A}^{-1}(\tilde{\mathbf{c}} - \mathbf{a}_1) &= 1 \\
\tilde{\mathbf{b}}^\top \tilde{A}^{-1}\tilde{\mathbf{c}}^k &= 1, \quad k = 2, \dots, p, \\
\tilde{\mathbf{b}}^\top \left[\tilde{A}^{-k}(\tilde{\mathbf{c}} - \mathbf{a}_1) - \tilde{A}^{-k+1}\tilde{\mathbf{e}}\right] &= 0, \quad k = 2, \dots, p, \\
\tilde{\mathbf{b}}^\top \tilde{A}^{-l}\left[\tilde{A}^{-1}\tilde{\mathbf{c}}^{k-l} - (k-l)\tilde{\mathbf{c}}^{k-l-1}\right] &= 0,
\end{aligned}
$$

for $k = 2, \dots, \infty$ and $l = \max\{1, k-p\}, \dots, k-1$.

Remarks

- If the ESDIRK method is stiffly accurate conditions the first two conditions are automatically satisfied.
- If the ESDIRK method is consistent and satisfies $C(q)$ the third condition is automatically satisfied for $k = 2, \dots, \infty$ and $l = \max\{1, k-q\}, \dots, k-1$.

New methods

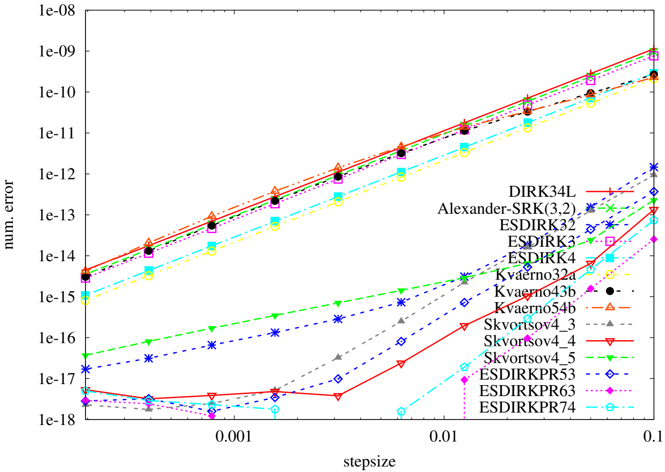
Methods for index-2 DAEs

- ESDIRK32: $s = 4$, $p = 2$ (Williams, Burrage, Cameron and Kerr, 2002
- Skvortsov4-3: $s = 6$, $p = 4$ (Skvortsov, 2010)
- Skvortsov4-4: $s = 6$, $p = 4$ (Skvortsov, 2010)
- Skvortsov4-5: $s = 8$, $p = 5$ (Skvortsov, 2010)

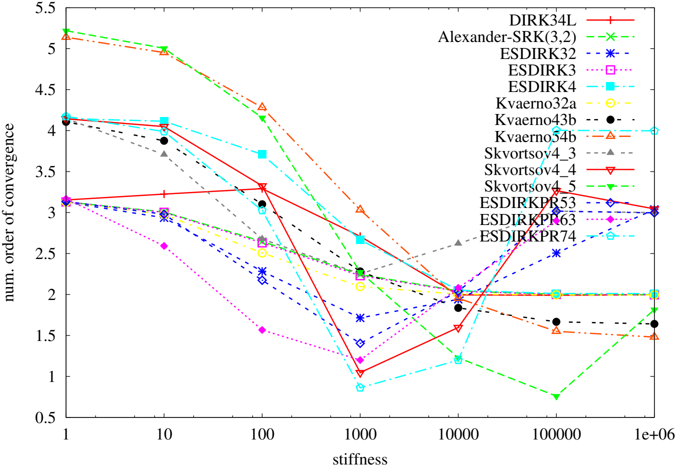
Methods for PR

- ESDIRKPR53: $s = 5$, $p = 3$
- ESDIRKPR63: $s = 6$, $p = 3$
- ESDIRKPR74: $s = 7$, $p = 4$

Example of Prothero and Robinson ($\lambda = -10^6$)



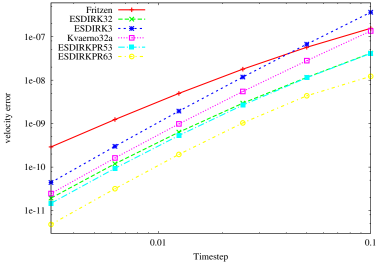
Numerical order of convergence

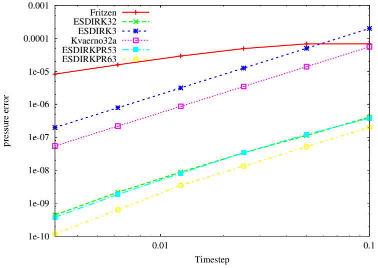


Navier–Stokes equations (3rd order methods)

Exact solution:

$$\begin{aligned} u_1(t,x,y) &= t^3y^2, \\ u_2(t,x,y) &= t^2x, \\ p(t,x,y) &= tx+y-(t+1)/2 \end{aligned}$$

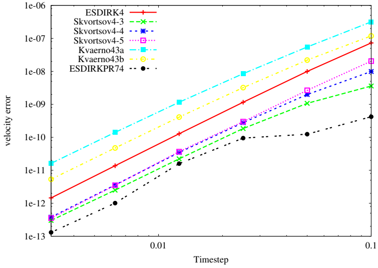


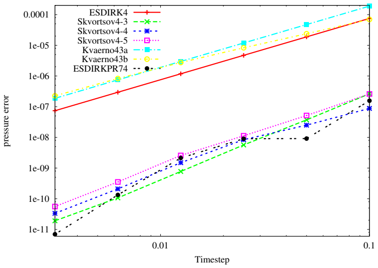


Navier–Stokes equations (4th order methods)

Exact solution:

$$u_1(t,x,y) = t^3y^2,$$
$$u_2(t,x,y) = t^2x,$$
$$p(t,x,y) = tx + y - (t+1)/2$$





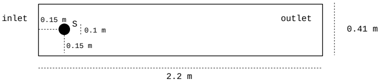
2D Benchmark problem

$J = (0, 8)$

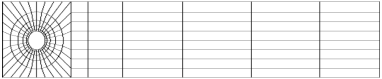
Problem:

$$\begin{cases} \dot{u} - Re^{-1}\Delta u & \\ +(u \cdot \nabla)u + \nabla p & = f \\ \nabla \cdot u & = 0 \end{cases}$$

Domain:



Coarsest grid:



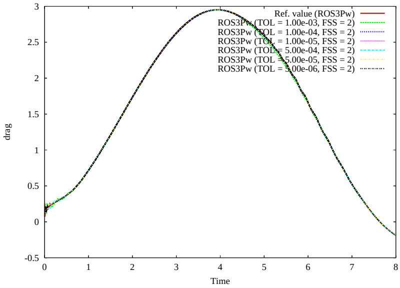
boundary conditions:

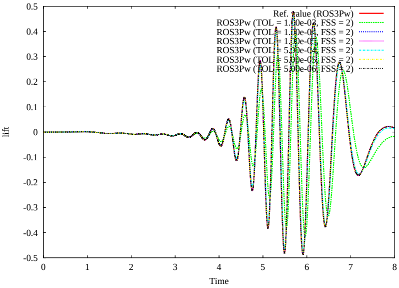
$$u(t, 0, y) = u(t, 2.2, y) = 0.41^{-2} \sin(\pi t/8)(6y(0.41-y), 0)\, m\, s^{-1}, \; 0 \leqslant y \leqslant 0.41.$$

On all other boundaries: $u = 0$

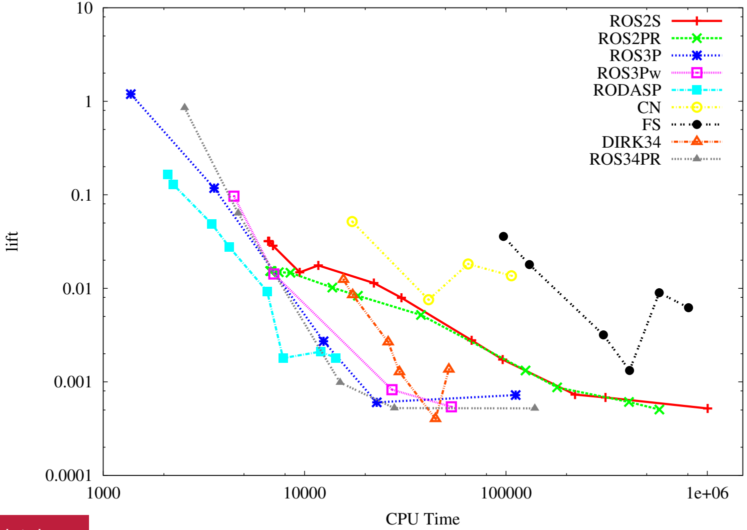
Drag and lift coefficient

$$c_d(t) = -20\left[(\mathbf{u}_t, \mathbf{v}_d) + (\nu\nabla\mathbf{u}, \nabla\mathbf{v}_d) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}_d) - (p, \nabla\cdot\mathbf{v}_d)\right],$$
$$c_l(t) = -20\left[(\mathbf{u}_t, \mathbf{v}_l) + (\nu\nabla\mathbf{u}, \nabla\mathbf{v}_l) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}_l) - (p, \nabla\cdot\mathbf{v}_l)\right]$$





The lift coefficient



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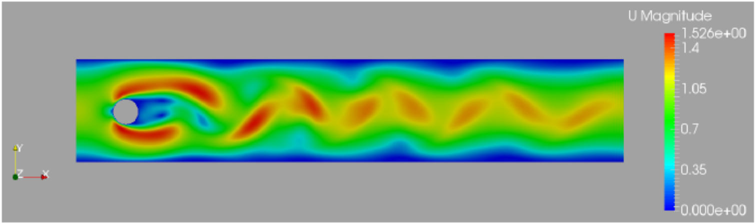
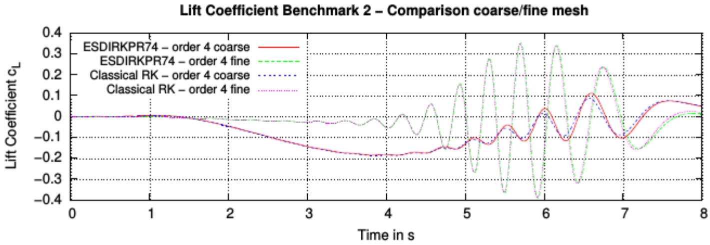
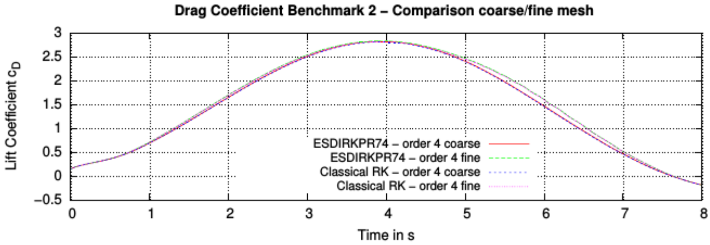
Velocity field

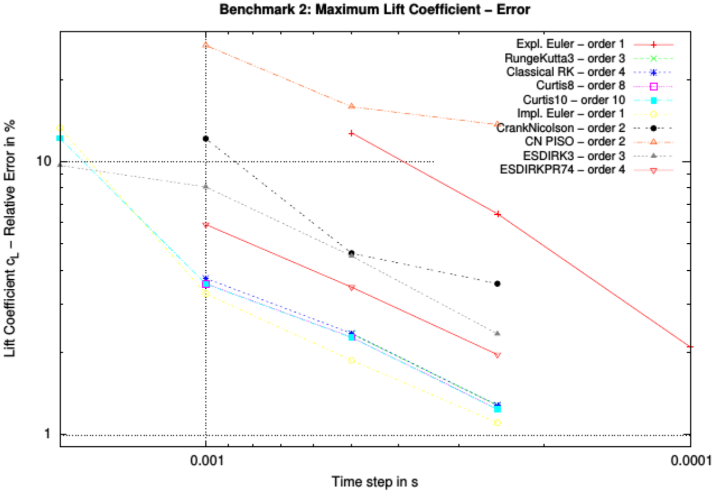


Drag and lift coefficient





Performance



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Summary and Outlook

- DIRK methods have stages order $\leqslant 2 \implies$ order reduction for the Prothero–Robinson example and NSE
- Error analysis for the Prothero–Robinson example $\implies$ further order conditions
- Solution of the nonlinear systems is expensive $\implies$ use linear implicit RK methods, i.e. Rosenbrock–Wanner methods
- Again further order conditions must be satisfied to avoid order reduction
- For higher order ($p > 4$) fully implicit RK methods can be applied. If the simplified Newton method and a transformation of the coefficient matrix A is used, the linear systems split and can be solved in parallel.

Literature

- Student work: J. Wiegmann: On the Implementation of Explicit Runge-Kutta and ESDIRK Methods for Laminar Incompressible Transient Flows Using OpenFOAM, Student work, TU Braunschweig, 2016.
- Papers:
  - An analysis of the Prothero-Robinson example for constructing new DIRK and ROW methods. Journal of Computational and Applied Mathematics (262), pp. 105-114, 2012
  - Apdative timestep control for fully implicit Runge–Kutta methods of higher order. Informatik-Bericht 2014-03, TU BS, 2014
  - The Prothero and Robinson example: Convergence studies for Runge–Kutta and Rosenbrock–Wanner methods Informatik-Bericht 2014-08, TU BS, 2014
  - Improved traditional Rosenbrock–Wanner methods for stiff ODEs and DAEs. Journal of Computational and Applied Mathematics, volume 286, pages 128–144, 2015
  - An analysis of the Prothero–Robinson example for constructing new adaptive ESDIRK methods of order 3 and 4. Applied Numerical Mathematics, volume 94, pages 75–87, 2015
  - A component framework for the parallel solution of the incompressible Navier-Stokes equations with Radau-IIA methods (with R. Niekamp), Int. J. Numer. Meth. Fluids, DOI: 10.1002/fld.4018, 2015